Identifiability and Stability in Blind Deconvolution under Minimal Assumptions

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Blind Deconvolution

\[ u \ast v = z \]

Both \( u \) and \( v \) are unknown \( \implies \) **Ill-posed bilinear inverse problem**

- Solved with “good” priors (e.g., subspace, sparsity)

- **Empirical success in various applications** (e.g., blind image deblurring, speech dereverberation, seismic data analysis, etc.)
  - Theoretical results are limited. \( \implies \) **The focus of this presentation**
Problem Statement

- Signal: $u_0 \in \mathbb{C}^n$
- Filter: $v_0 \in \mathbb{C}^n$
- Measurement: $z = u_0 \ast v_0 \in \mathbb{C}^n$

Find $(u, v)$

s.t. $u \ast v = z,$

$u \in \Omega_U, v \in \Omega_V.$

Three scenarios:

1. Subspace constraints
2. Sparsity constraints
3. Mixed constraints
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- **Signal:** $u_0 \in \mathbb{C}^n$
- **Filter:** $v_0 \in \mathbb{C}^n$
- **Measurement:** $z = u_0 \otimes v_0 \in \mathbb{C}^n$

Find $(u, v)$ such that

$$u \otimes v = z,$$

$$u \in \Omega_U, \ v \in \Omega_V.$$

Three scenarios:

1. Subspace constraints
2. Sparsity constraints
3. Mixed constraints
Problem Statement

- **Signal:** \( u_0 = Dx_0 \), the columns of \( D \in \mathbb{C}^{n \times m_1} \) form a basis or a frame
- **Filter:** \( v_0 = Ey_0 \), the columns of \( E \in \mathbb{C}^{n \times m_2} \) form a basis or a frame
- **Measurement:** \( z = u_0 \ast v_0 = (Dx_0) \ast (Ey_0) \in \mathbb{C}^{n} \)

\[
\text{(BD) Find } (x, y) \\
\text{s.t. } (Dx) \ast (Ey) = z, \\
x \in \Omega_X, y \in \Omega_Y.
\]

Three scenarios:

1. **Subspace constraints:**
   \( \Omega_X = \mathbb{C}^{m_1} \) and \( \Omega_Y = \mathbb{C}^{m_2} \)

2. **Sparsity constraints:**
   \( \Omega_X = \{ x \in \mathbb{C}^{m_1} : \|x\|_0 \leq s_1 \} \) and \( \Omega_Y = \{ y \in \mathbb{C}^{m_2} : \|y\|_0 \leq s_2 \} \)

3. **Mixed constraints:**
   \( \Omega_X = \{ x \in \mathbb{C}^{m_1} : \|x\|_0 \leq s_1 \} \) and \( \Omega_Y = \mathbb{C}^{m_2} \)
Weak and Strong Identifiability

**Definition (Identifiability up to scaling)**

- **Weak identifiability (\((x_0, y_0)\) is identifiable):** every solution \((x, y)\) satisfies \(x = \sigma x_0\) and \(y = \frac{1}{\sigma} y_0\) for some nonzero scalar \(\sigma\).

- **Strong identifiability (\(\Omega_X \times \Omega_Y\) is identifiable):** every \((x_0, y_0) \in \Omega_X \times \Omega_Y\) is identifiable up to scaling.

**Lifting**

Define \(G_{DE} : \mathbb{C}^{m_1 \times m_2} \to \mathbb{C}^n\) such that \(G_{DE}(xy^T) = (Dx) \odot (Ey)\), and \(M_0 = x_0y_0^T \in \Omega_M = \{xy^T : x \in \Omega_X, y \in \Omega_Y\}\).

(BD) Find \((x, y)\), s.t. \((Dx) \odot (Ey) = z\), \(x \in \Omega_X, y \in \Omega_Y\).

(Lifted BD) Find \(M\), s.t. \(G_{DE}(M) = z\), \(M \in \Omega_M\).

- Weak identifiability \(\iff\) Unique recovery of \(M_0\)
- Strong identifiability \(\iff\) Uniform unique recovery of all matrices in \(\Omega_M\)
Weak and Strong Identifiability

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- **Strong identifiability** ($\Omega_X \times \Omega_Y$ is identifiable): every $(x_0, y_0) \in \Omega_X \times \Omega_Y$ is identifiable up to scaling.

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Define $G_{DE}: \mathbb{C}^{m_1 \times m_2} \rightarrow \mathbb{C}^n$ such that $G_{DE}(xy^T) = (Dx) \otimes (Ey)$, and $M_0 = x_0 y_0^T \in \Omega_M = \{xy^T : x \in \Omega_X, y \in \Omega_Y\}$.

**(BD) Find** $(x, y)$, s.t. $(Dx) \otimes (Ey) = z$, $x \in \Omega_X, y \in \Omega_Y$.  

**(Lifted BD) Find** $M$, s.t. $G_{DE}(M) = z$, $M \in \Omega_M$.  

- Weak identifiability $\iff$ Unique recovery of $M_0$  
- Strong identifiability $\iff$ Uniform unique recovery of all matrices in $\Omega_M$
Single-point and Uniform Stability

\[
\text{min}_M \| G_{DE}(M) - z \|_2 , \quad \text{s.t. } M \in \Omega_B := \Omega_M \bigcap \mathcal{B}_{C^{m_1 \times m_2}}.
\]

\[\text{Definition (Stability)}\]

\text{Single-point stability at } M_0: \| G_{DE}(M) - G_{DE}(M_0) \|_2 \leq \delta \text{ for } M \in \Omega_B, \text{ only if } \| M - M_0 \|_2 \leq \varepsilon.

\text{Uniform Stability on } \Omega_B: \| G_{DE}(M_1) - G_{DE}(M_2) \|_2 \leq \delta \text{ for } M_1, M_2 \in \Omega_B, \text{ only if } \| M_1 - M_2 \|_2 \leq \varepsilon.

- Strong identifiability + single-point stability at \( M_0 \Rightarrow G_{DE}^{-1} \) is continuous at \( G_{DE}(M_0) \)
- Uniform stability on \( \Omega_B \Rightarrow G_{DE}^{-1} \) is uniformly continuous on \( \Omega_B \)
- Stability \( \Rightarrow \) solution to (Noisy BD) is accurate
Main Results: Identifiability

Sample complexity constant $d =$
- $m_1 + m_2$ – subspace constraints
- $s_1 + m_2$ – mixed constraints
- $s_1 + s_2$ – sparsity constraints

Theorem (Identifiability)

- If $n > d$, then we have weak identifiability for almost all $D \in \mathbb{C}^{n \times m_1}$ and $E \in \mathbb{C}^{n \times m_2}$.
- If $n > 2d$, then we have strong identifiability for almost all $D \in \mathbb{C}^{n \times m_1}$ and $E \in \mathbb{C}^{n \times m_2}$.

The same sample complexities hold for the case where $x, y, D, E$ are real!
Main Results: Stability

Theorem (Stability)

\( D \in \mathbb{C}^{n \times m_1} \) and \( E \in \mathbb{C}^{n \times m_2} \) independent random, s.t.: 
\[
\{(FD)(j,:)\}^n_{j=1} \sim \text{i.i.d. uniform distribution on } RB_{\mathbb{C}^{m_1}}
\]
\[
\{(FE)(j,:)\}^n_{j=1} \sim \text{i.i.d. uniform distribution on } RB_{\mathbb{C}^{m_2}}
\]

- If \( n > d \), then we have single-point stability w.p. at least 
  \[ 1 - C' \left( \frac{\delta^2}{R^4} \right)^{n-d} \left( \frac{1}{\varepsilon^2} \right)^n. \]

- If \( n > 2d \), then we have uniform stability w.p. at least 
  \[ 1 - C'' \left( \frac{\delta^2}{R^4} \right)^{n-2d} \left( \frac{1}{\varepsilon^2} \right)^n. \]

<table>
<thead>
<tr>
<th>Constrain Type</th>
<th>( d )</th>
<th>( C' )</th>
<th>( C'' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subspace constraints</td>
<td>( m_1 + m_2 )</td>
<td>( \frac{C^n}{n(n-d)} )</td>
<td>( \frac{(4C)^n}{n(n-2d)} )</td>
</tr>
<tr>
<td>Mixed constraints</td>
<td>( s_1 + m_2 )</td>
<td>( \left( \frac{m_1}{s_1} \right)^2 \frac{C^n}{n(n-d)} )</td>
<td>( \left( \frac{m_1}{s_1} \right)^4 \frac{(4C)^n}{n(n-2d)} )</td>
</tr>
<tr>
<td>Sparsity constraints</td>
<td>( s_1 + s_2 )</td>
<td>( \left( \frac{m_1}{s_1} \right)^2 \left( \frac{m_2}{s_2} \right)^2 \frac{C^n}{n(n-d)} )</td>
<td>( \left( \frac{m_1}{s_1} \right)^4 \left( \frac{m_2}{s_2} \right)^4 \frac{(4C)^n}{n(n-2d)} )</td>
</tr>
</tbody>
</table>

Here, \( C = 648 m_1 m_2 \left( 1 + 2 \ln \frac{2\sqrt{\pi R^2}}{3\delta} \right) \).

Similar stability results hold for the case where \( x, y, D, E \) are real!
Main Results: Summary

- \[ \text{RSNR} = \frac{\|M_0\|_2^2}{\|M - M_0\|_2^2}, \quad \text{MSNR} = \frac{\|G_{DE}(M_0)\|_2^2}{\|G_{DE}(M) - G_{DE}(M_0)\|_2^2}. \]

The probability of failure (unstable reconstruction) is roughly \( \text{RSNR}^n \cdot \text{MSNR}^{-(n-d)} \).

- Identifiability for almost all \( D, E \) \( \Longrightarrow \) Unique recovery for random \( D, E \) w.p. 1
  Stability for random \( D, E \) w.h.p. \( \Longrightarrow \) Unique recovery for random \( D, E \) w.p. 1

- Identifiability on a cone constraint set v.s. stability on the cone restricted to a ball.
  From a practical point of view, because the radius can be arbitrarily large, this restriction is of no significant consequence.
Lifting: BD as a Matrix Recovery Problem

Frequency domain measurement:

$$\tilde{z}^{(j)} := \frac{1}{\sqrt{n}} (F z)^{(j)} = (FD)^{(j,:)} x_0 (FE)^{(j,:)} y_0 + \frac{1}{\sqrt{n}} (Fe)^{(j)} = a_j^* M_0 b_j + \tilde{e}^{(j)} ,$$

where $M_0 = x_0 y_0^T$, $a_j = (FD)^{(j,:)*}$, $b_j = (FE)^{(j,:)*}$, and $\tilde{e} = \frac{1}{\sqrt{n}} Fe$. Define

$$A(M) = \begin{bmatrix} a_1^* M b_1, a_2^* M b_2, \cdots, a_n^* M b_n \end{bmatrix}^T .$$

Then

$$A(M) = \frac{1}{\sqrt{n}} F G_{DE}(M), \quad \text{and} \quad \|A(M)\|_2 = \frac{1}{\sqrt{n}} \|G_{DE}(M)\|_2 .$$

Next, we study unique / stable matrix recovery!
Unique and Stable Matrix Recovery

**Theorem (Unique Matrix Recovery)**

For almost all $a_j \in \mathbb{C}^{m_1}$ and $b_j \in \mathbb{C}^{m_2}$ ($j = 1, 2, \cdots, n$):

- the recovery of $M_0$ from $\tilde{z} = A(M_0)$ is unique if $n > d$.
- the recovery of all $M \in \Omega_M$ from $\tilde{z} = A(M)$ is unique if $n > 2d$.

**Theorem (Single-point Stable Matrix Recovery, Subspace Constraints)**

\[
\{a_j\}_{j=1}^n \overset{i.i.d.}{\sim} \text{Uniform}(RB_{\mathbb{C}^{m_1}}), \text{ and } \{b_j\}_{j=1}^n \overset{i.i.d.}{\sim} \text{Uniform}(RB_{\mathbb{C}^{m_2}}).
\]

If $n > m_1 + m_2$, then, with probability at least

\[
1 - \left(648 m_1 m_2 \left(1 + 2 \ln \frac{2R^2}{3\delta}\right)\right)^n \left(\frac{\delta^2}{R^4}\right)^{n-m_1-m_2} \left(\frac{1}{\varepsilon^2}\right)^n,
\]

we have $\|A(M) - A(M_0)\|_2 \leq \delta$ only if $\|M - M_0\|_2 \leq \varepsilon$. 

Proof Sketch

Two main ingredients:
- The constraint set $\Omega_B$ is “small”: it has a small covering number
- The measurement vectors $\{a_j, b_j\}_{j=1}^n$ are “generic”: their probability distribution satisfies certain concentration of measure bounds

Definition (Covering Number)

For a nonempty bounded set $\Omega_B \subset \mathbb{C}^{m_1 \times m_2}$,

$$N_{\Omega_B}(\rho) := \min \left\{ N \in \mathbb{Z}^+ : \exists M_i \in \mathbb{C}^{m_1 \times m_2}, i = 1, 2, \cdots, N \right\}$$

s.t. $\Omega_B \subset \bigcup_{i \in \{1, 2, \cdots, N\}} (M_i + \rho \mathcal{B}_{\mathbb{C}^{m_1 \times m_2}})$.

Lemma (Covering Number Bound, Subspace Constraints)

$$N_{\Omega_B}(\rho) \leq \left( \frac{6\sqrt{2}}{\rho} \right)^{2m_1+2m_2} \text{ for all } 0 < \rho < 1.$$
Proof Sketch

Two main ingredients:

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Lemma (Concentration of Measure)

Independent $\{a_j\}_{j=1}^n \overset{i.i.d.}{\sim} \text{Uniform}(RB_{\mathbb{C}^m_1})$, $\{b_j\}_{j=1}^n \overset{i.i.d.}{\sim} \text{Uniform}(RB_{\mathbb{C}^m_2})$. If $\ell \leq \|M\|_2 \leq L$, then

$$\mathbb{P} \left[ \|a^* M \bar{b}\| \leq \rho \right] \leq \rho^2 g(\rho, \ell, L, R),$$

where $g(\rho, \ell, L, R)$ satisfies $\lim_{\rho \to 0} \frac{\log g(\rho, \ell, L, R)}{\log \frac{1}{\rho}} = 0$. 
Proof Sketch.

Define $\Omega_\varepsilon := \{ M \in \Omega_B - M_0 : \|M\|_2 > \varepsilon \}$, then

$$P_f \leq \mathbb{P}[\exists M \in \Omega_\varepsilon \text{ s.t. } \|A(M)\|_2 \leq \delta]$$

Form a minimal cover of $\Omega_\varepsilon$ with balls of radius $\frac{\delta}{R^2} < 1$

Replace $\Omega_\varepsilon$ with this cover, and apply a union bound

$$\|A(M)\|_2 \leq \delta \implies |\langle A_j, M \rangle| \leq \delta \ (\forall j \in [m]) \quad (|z_j| \leq \|z\|_2)$$

$$\implies |\langle A_j, M_c \rangle| \leq 3\delta \ (\forall j \in [m]) \quad \text{(triangle inequality)}$$
Summary

- First tight sample complexity bounds for unique and stable blind deconvolution (the bounds are optimal, to within a few samples).
- Identifiability results hold for generic bases or frames (invalid on a set of Lebesgue measure 0). If the bases/frames are drawn from a distribution absolutely continuous w.r.t. the Lebesgue measure, then the results hold w.p. 1.
- Stability results hold w.h.p. for specified distribution. In fact, the results can be generalized to a large class of distributions that satisfy certain concentration of measure properties.
- These results are fundamental to blind deconvolution, independent of algorithms.

Details:
https://arxiv.org/abs/1507.01308
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Thank you!